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FROM OPTIMAL CONTROL PROBLEMS TO FOLIATIONS, SOME KINDS OF MULTIFOLIATIONS AND RELATIONS TO GENERALISED JETS

(Daripada Masalah Kawalan Optimum kepada Foliasi, Beberapa Jenis Multifoliasi dan Hubungan dengan Jet Teritlak)

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ABSTRACT

Transversal foliations and jets modulo foliations are studied. It is shown that multifoliations provides a way for a more general description of (R, S, Q)-jets.

Keywords: Optimal control; regulation; foliation; multifoliation; transversality; jet; (R, S, Q)-jet

ABSTRAK

Foliasi rentas lintang dan jet modulo foliasi dikaji. Ditunjukkan bahawa multifoliasi membuka jalan untuk pemerihalan yang lebih umum bagi jet-(R, S, Q).

Kata kunci: Kawalan optimum; regulasi; foliasi; multifoliasi; kerentas-lintangan; jet; jet-(R, S, Q)

1. Introduction

This paper represents an extended version of the author's contribution (Kureš 2008) to the International Symposium on New Developments of Geometric Function Theory and its Applications, Bangi 2008. In particular, the first section is completely new and it represents a possible motivation for a study of foliations and multifoliations. It refers to fundamentals of geometric optimal control theory and it also refers to some classical examples (cf. Chapter 15 of La Valle (2006)) which are elaborated in detail here. The second section is focused to the study of foliations of smooth manifolds. We introduce concepts of $\cap$-transversality and $\cup$-transversality as we hope that this approach provides a more precise view to the transversality in itself and gives a good arrangement of various multifoliated structures. Further, we initialise jet formalism for such multifoliated structures. We have two main inspirations here: Ikegami's paper (Ikegami 1986) about jets modulo foliations and the concept of (R, S, Q)-jet, see e.g. Kolář *et al.* (1993) or Doupovec & Kolář (1999). We present a way to a generalisation and unification (in a way) of both these jet languages.

We remark that multifoliations can be meaningfully used just in the optimal control. Indeed, partitions of manifolds induced by integrable distributions even satisfy some of transversality conditions (introduced in Section 2) in a number of practical situations.

As usually, all manifolds and maps are assumed to be smooth, i.e. of class C^∞.

2. Possible motivations for study of foliations by differential geometry

2.1. *Lie bracket*

Let M be a smooth manifold, $\dim M = m$, for an initial simplification it suffices think about $M = \mathbb{R}^m$. Points of M are written as

$$x = (x^1, \dots, x^m), \quad x_0 = (x_0^1, \dots, x_0^m), \quad \text{etc.} \tag{1}$$

We consider a smooth map $\gamma\colon I \to M$, where I is an interval in $\mathbb{R}$, usually containing 0. Such a map is called *a (smooth) curve* in M. Its equations[1] are

$$x^i = \gamma^i(t), \quad i = 1, \dots, m. \tag{2}$$

Let $\gamma(0) = x_0$, then $\frac{\mathrm{d}\gamma}{\mathrm{d}t}(0)$ determines a *tangent vector* to γ in x_0 with coordinates

$$\left(x_0^1, \dots, x_0^m, y_0^1, \dots, y_0^m\right) = \left(\gamma^1(0), \dots, \gamma^m(0), \frac{\mathrm{d}\gamma^1}{\mathrm{d}t}(0), \dots, \frac{\mathrm{d}\gamma^m}{\mathrm{d}t}(0)\right). \tag{3}$$

Tangent vectors to all curves γ going through x_0 form a m-dimensional vector space $T_{x_0}M$, *tangent space in* x_0. The vector coordinates are

$$\left(x_0^1, \dots, x_0^m, y^1, \dots, y^m\right). \tag{4}$$

We define the *tangent bundle* TM by the (disjoint) union

$$TM = \bigcup_{x_0 \in M} T_{x_0}M; \tag{5}$$

TM is $2m$-dimensional as a manifold with coordinates

$$\left(x^1, \dots, x^m, y^1, \dots, y^m\right) = (x, y). \tag{6}$$

We have a canonical projection $\pi\colon TM \to M$ sending (x, y) to (x). The *vector field* on M is a smooth section $X\colon M \to TM$, i.e. such a smooth map, for which $\pi \circ X = \mathrm{id}_M$. In coordinates,

$$X\colon \left(x^1, \dots, x^m\right) \mapsto \left(x^1, \dots, x^m, \xi^1(x), \dots, \xi^m(x)\right) \tag{7}$$

and $X(x_0)$ is (for a fixed point x_0) a tangent vector as in (3). If $f\colon M \to \mathbb{R}$ is a smooth function and X a vector field on M, we define a new function $Xf\colon M \to \mathbb{R}$ called the *Lie derivative* of f along X by

$$Xf(x_0) = \frac{\partial f}{\partial x^i}\xi^i(x_0). \tag{8}$$

We also write $Xf = \frac{\partial f}{\partial x^i}\xi^i$ and $X = \xi^i \frac{\partial}{\partial x^i}$. The vector field X is called *smooth*, if Xf is smooth for every smooth function f. If X is a given vector field on M, then the integral curve of X is such a curve γ which has a tangent vector in every its point x_0 equal to $X(x_0)$, i.e.

$$\left(\frac{\mathrm{d}\gamma^1}{\mathrm{d}t}, \dots \frac{\mathrm{d}\gamma^m}{\mathrm{d}t}\right) = \left(\xi^1(\gamma(t)), \dots, \xi^m(\gamma(t))\right); \tag{9}$$

[1] for a general M ($\neq \mathbb{R}^m$) we have local expressions in local coordinates

for a finding of integral curves is necessary to solve a system of m first order ordinary differential equations

$$\dot{\gamma} = X(\gamma). \tag{10}$$

In particular, we apply *maximal integral curves*: such a integral curve can not be a proper subset of another integral curve. If $\gamma_x \colon I \to M$ is a maximal integral curve of X satisfying $\gamma_x(0) = x$, then, by

$$\mathrm{Fl}^X_t(x) = \mathrm{Fl}^X(t, x) = \gamma_x(t), \tag{11}$$

are defined maps

$$\mathrm{Fl}^X_t \colon M \to M \quad \text{and} \quad \mathrm{Fl}^X \colon I \times M \to M. \tag{12}$$

The map Fl^X is called the *flow* [2] of the vector field X. Evidently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Fl}^X_t(x) = X\left(\mathrm{Fl}^X_t(x)\right) \tag{13}$$

holds. The important property of the flow is

$$\mathrm{Fl}^X(t+s, x) = \mathrm{Fl}^X\left(t, \mathrm{Fl}^X(s, x)\right). \tag{14}$$

A real vector space V is called the *Lie algebra*, if it contains a binary operation named the *bracket* and denoting by $[,]$ having the following properties:

1. $(u, v) \mapsto [u, v]$ is bilinear,

2. $(u, v) \mapsto [u, v]$ is antisymmetric,

3. *Jacobi identity* $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$ holds.

We define the *Lie bracket* $[X, Y]$ of vector fields $X = \xi^i \frac{\partial}{\partial x^i}$, $Y = \eta^i \frac{\partial}{\partial^i}$ by

$$[X, Y]f = X(Yf) - Y(Xf), \tag{15}$$

it follows coordinates of $Z = [X, Y] = \zeta^i \frac{\partial}{\partial x^i}$ are

$$\zeta^i = \xi^j \frac{\partial \eta^i}{\partial x^j} - \eta^j \frac{\partial \xi^i}{\partial x^j} \tag{16}$$

and this bracket provides the structure of Lie algebra to every vector space $T_{x_0}M$. We say that vector fields X, Y *commute*, if $[X, Y] = 0$.

[2] the flow maps the point x to another point lying just on the maximal curve going through x; viewing this curve as a motion, x has time 0 and its image a time t

2.2. *Why is the Lie bracket so important in optimal control*

The system (10) can be written as

$$\dot{x}^i = \xi^i(x) = \xi^i(x^1,\dots,x^m), \quad i = 1,\dots,m. \tag{17}$$

In optimal control, we investigate the system

$$\dot{x}^i = \phi^i(x,u) = \phi^i(x^1,\dots,x^m,u^1,\dots,u^k), \quad i = 1,\dots,m, \tag{18}$$

where $u^1(t),\dots,u^k(t)$ is so-called *regulation*[3]. The *linear system* has a form

$$\dot{x}^i = A^i_j x^j + B^i_p u^p, \quad i = 1,\dots,m, \tag{19}$$

otherwise we have *nonlinear systems*. A frequent nonlinear system is so-called *affine system*

$$\dot{x} = X_p(x)u^p + X_0(x), \quad p = 1,\dots,k \tag{20}$$

where $X_1,\dots,X_k$, X_0 are vector fields. For $X_0 = 0$, we talk about the *driftless affine system* (X_0 is called the *drift term*). In particular, let us consider a driftless affine optimal control system

$$\dot{x} = X(x)u^1 + Y(x)u^2, \text{ where } u(t) = \big(u^1(t),u^2(t)\big) = \begin{cases} (1,0) & \text{for } t \in [0,\epsilon] \\ (0,1) & \text{for } t \in [\epsilon,2\epsilon] \\ (-1,0) & \text{for } t \in [2\epsilon,3\epsilon] \\ (0,-1) & \text{for } t \in [3\epsilon,4\epsilon] \end{cases} \tag{21}$$

We compute Taylor expansions for the first part of the motion as

$$\begin{aligned} x(\epsilon) &= x(0) + \epsilon\dot{x}(0) + \frac{1}{2}\epsilon^2\ddot{x}(0) + \mathcal{O}(\epsilon^3) \\ &= x(0) + \epsilon X(x(0)) + \frac{1}{2}\epsilon^2\frac{\partial X}{\partial x}(x(0))X(x(0)) + \mathcal{O}(\epsilon^3), \end{aligned} \tag{22}$$

and for the second part as

$$\begin{aligned} x(2\epsilon) &= x(\epsilon) + \epsilon\dot{x}(\epsilon) + \frac{1}{2}\epsilon^2\ddot{x}(\epsilon) + \mathcal{O}(\epsilon^3) \\ &= x(\epsilon) + \epsilon Y(x(\epsilon)) + \frac{1}{2}\epsilon^2\frac{\partial Y}{\partial x}(x(\epsilon))Y(x(\epsilon)) + \mathcal{O}(\epsilon^3), \end{aligned} \tag{23}$$

in which we substitute $x(\epsilon)$ from (22) and use the fact that for the infinitesimal ϵ the relation $Y(x(0)+X(x(0))) = \epsilon\frac{\partial Y}{\partial x}(x(0))X(x(0))$ holds [4]; we obtain

$$\begin{aligned} x(2\epsilon) &= x(0) + Y(x(0) \\ &+ \epsilon^2\left(\frac{1}{2}\frac{\partial X}{\partial x}(x(0))X(x(0)) + \frac{\partial Y}{\partial x}(x(0))X(x(0)) + \frac{1}{2}\frac{\partial Y}{\partial x}(x(0))Y(x(0))\right) \\ &+ O(\epsilon^3). \end{aligned} \tag{24}$$

[3] see the next section

[4] this fact is a slight generalisation of the well-known expression $f(x_0+\epsilon) \approx \epsilon f'(x_0)$ for a function f and a point x_0

The same process is used for $x(3\epsilon)$ and $x(4\epsilon)$, the final result is

$$x(4\epsilon) = x(0) + \epsilon^2 \left(\frac{\partial Y}{\partial x}(x(0))X(x(0)) - \frac{\partial X}{\partial x}(x(0))Y(x(0)) \right) + O(\epsilon^3). \tag{25}$$

The computation shows that, at each point, infinitesimal motion is possible not only in the directions contained in the span of the input vector fields X and Y, but also in the directions of their Lie bracket $[X, Y] = \frac{\partial Y}{\partial x} X - \frac{\partial X}{\partial x} Y$. It is also possible to obtain motion in the direction of higher-order brackets, such as $[X, [X, Y]]$, $[Y, [X, Y]]$, etc.

2.3. *A note about regulations*

For the regulation $u\colon \mathbb{R} \to \mathbb{R}^k$, $u(t) = \big(u^1(t), \dots, u^k(t)\big)$ are not any requirements for a smoothness nor even for a continuity of functions $u^p\colon \mathbb{R} \to \mathbb{R}$, $p = 1, \dots, k$. Hereafter regulations are considered to be piecewise constant functions, it means there is a partition of the time line into intervals and in every such an interval J is

$$u(t) = c = (c^1, \dots, c^k), \quad t \in J,\ c^p, p = 1, \dots, k, \text{ are real constants.} \tag{26}$$

(Cf. (21) as an example.) Moreover, only some special subclasses of piece-wise constant functions are considered. Then we talk about *admissible regulations* and the subclass of admissible regulations is denoted by $\mathfrak{U}$, so we write $u \in \mathfrak{U}$.

2.4. *Distributions and foliations*

Suppose that for each $x_0 \in M$ and $k < m$ a k-dimensional vector subspace $D_{x_0}M$ of $T_{x_0}M$. Let us consider the (disjoint) union

$$DM = \bigcup_{x_0 \in M} D_{x_0}M. \tag{27}$$

The k-*distribution* on M is a smooth section $D\colon M \to DM$, i.e. such a map, which assigns to each point x_0 such a k-dimensional subspace. This is possible by such a way that

$$D(x) = \operatorname{span}\{X_1(x), \dots, X_k(x)\}, \tag{28}$$

where $X_1, \dots, X_k$ are linear independent smooth vector fields. Nevertheless, if arbitrary vector fields $X_1, \dots, X_k$ are given, then a dimension of $\operatorname{span}\{X_1(x), \dots, X_k(x)\}$ can be less than k and can differ in different points. (Such a map is counted as a *distribution*, not a k-distribution.) Let D be a k-distribution on M and N a n-dimensional submanifold of M, $n \leq k$. Then N is said an *integral manifold* of D, if $T_{x_0}N \subseteq D(x_0)$. An integral manifold of D is called the *maximal integral manifold* if it is not contained in any strictly larger integral manifold of D. The k-distribution D on M is called an *integrable distribution*, if each point of M is contained in some integral manifold of D. Then each point is contained in a unique maximal integral manifold, so the maximal integral manifolds form a partition of M. This partition is called the *foliation of M induced by the integrable distribution D*, and each maximal integral manifold is called a *leaf* of this foliation. Further, we say that a vector field X *lies* in D, if $X(x_0) \in D(x_0)$ for all $x_0 \in M$. If X lies in D, then integral

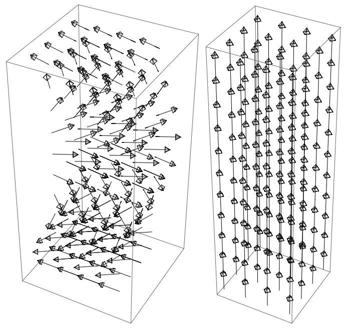

Figure 1: The vector fields X and Y

curve of X going through x_0 stays in the leaf through x_0. If $[X, Y]$ lies in D for any X, Y lying in D, we say that D is an *involutive distribution.* Now, we give the famous theorem on the geometry of distributions.

Frobenius Theorem. *D is an integrable distribution if and only if D is an involutive distribution.*

2.5. *A simplified model for differential drives and cars*

Let us consider a simplified model for differential drives and cars, i.e. a driftless affine system in $M = \mathbb{R}^3$ of a form

$$\begin{aligned} \dot{x} &= u^1 \cos\theta \\ \dot{y} &= u^1 \sin\theta \\ \dot{\theta} &= u^2. \end{aligned} \tag{29}$$

Thus, $X = (\cos\theta, \sin\theta, 0)$, $Y = (0, 0, 1)$.

We obtain integral curves of X and Y going through $[x_0, y_0, \theta_0]$ by the solving of systems

$$\begin{aligned} \dot{x} &= \cos\theta \\ \dot{y} &= \sin\theta \\ \dot{\theta} &= 0 \end{aligned} \tag{30}$$

having the solution

$$x = (\cos\theta_0)t + x_0, \quad y = (\sin\theta_0)t + y_0, \quad \theta = \theta_0 \tag{31}$$

and

$$\begin{aligned} \dot{x} &= 0 \\ \dot{y} &= 0 \\ \dot{\theta} &= 1 \end{aligned} \tag{32}$$

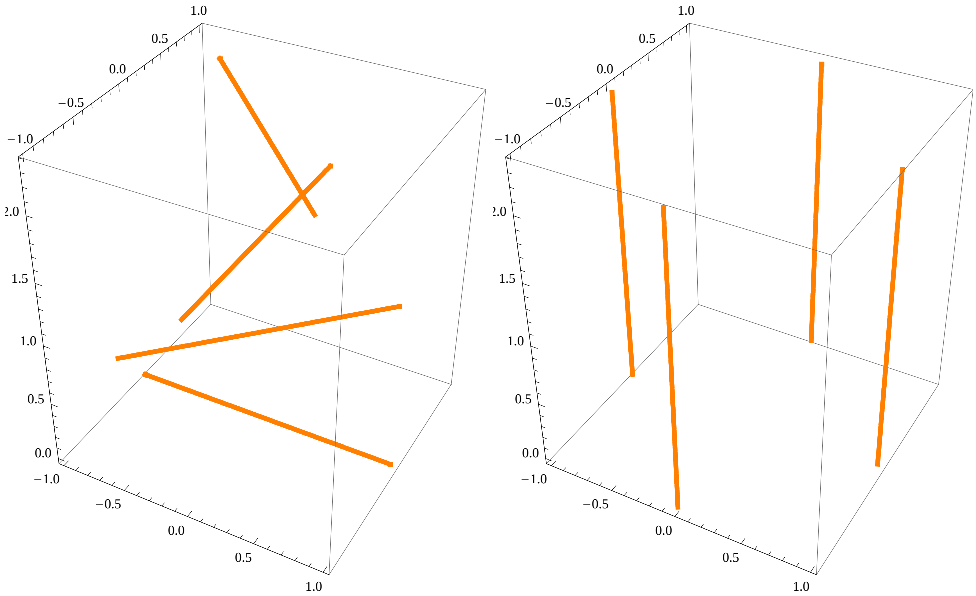

Figure 2: The integral curves of X and Y

having the solution

$$x = x_0, \quad y = y_0, \quad \theta = t + \theta_0. \tag{33}$$

The distribution D maps every point to a plane given by this point and by direction vectors of lines above:

$$D\colon (x_0, y_0, \theta_0) \mapsto \{(x_0, y_0, \theta_0) + u^1(\cos\theta_0, \sin\theta_0, 0) + u^2(0, 0, 1); u^1, u^2 \in \mathbb{R}\} \tag{34}$$

The distribution is a 2-distribution, because it is 2-dimensional in every point. Nevertheless,

$$[X, Y] = (\sin\theta, -\cos\theta, 0) \tag{35}$$

and this vector field is not a linear combination of X and Y . Hence $[X, Y]$ does not lie in D and D is not involutive. By Frobenius theorem, D is not integrable.

2.6. *Trapped on a sphere*

Let us consider a system in $M = \mathbb{R}^3$ of a form

$$\begin{aligned} \dot{x} &= u^1 y + u^2 z \\ \dot{y} &= u^1 x \\ \dot{z} &= -u^2 x. \end{aligned} \tag{36}$$

Thus, $X = (y, -x, 0)$, $Y = (z, 0, -x)$.

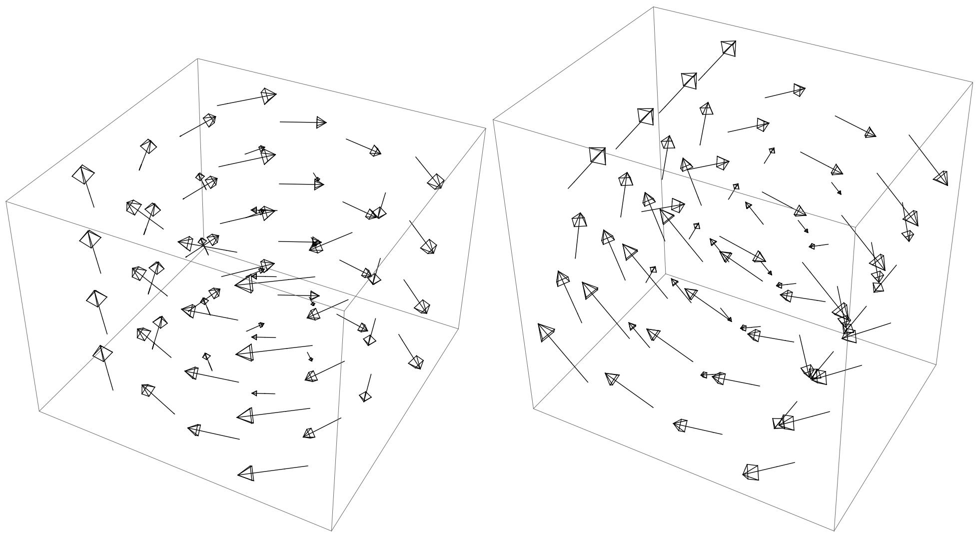

Figure 3: The vector fields X and Y

We obtain integral curves of X and Y going through $[x_0, y_0, z_0]$ by the solving of systems

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x \\ \dot{z} &= 0 \end{aligned} \tag{37}$$

having the solution

$$x = x_0 \cos t + y_0 \sin t, \quad y = -x_0 \sin t + y_0 \cos t, \quad z = z_0 \tag{38}$$

and

$$\begin{aligned} \dot{x} &= z \\ \dot{y} &= 0 \\ \dot{z} &= x \end{aligned} \tag{39}$$

having the solution

$$x = x_0 \cos t + z_0 \sin t, \quad y = y_0, \quad y = -x_0 \sin t + z_0 \cos t \tag{40}$$

The distribution D maps every point to a plane given by this point and by direction vectors of lines above:

$$D\colon (x_0, y_0, z_0) \mapsto \{(x_0, y_0, z_0) + u^1(y_0, -x_0, 0) + u^2(z_0, 0, -x_0); u^1, u^2 \in \mathbb{R}\} \tag{41}$$

The distribution is a 2-distribution, because it is 2-dimensional in every point. The Lie bracket is

$$[X, Y] = (0, z, -y) \tag{42}$$

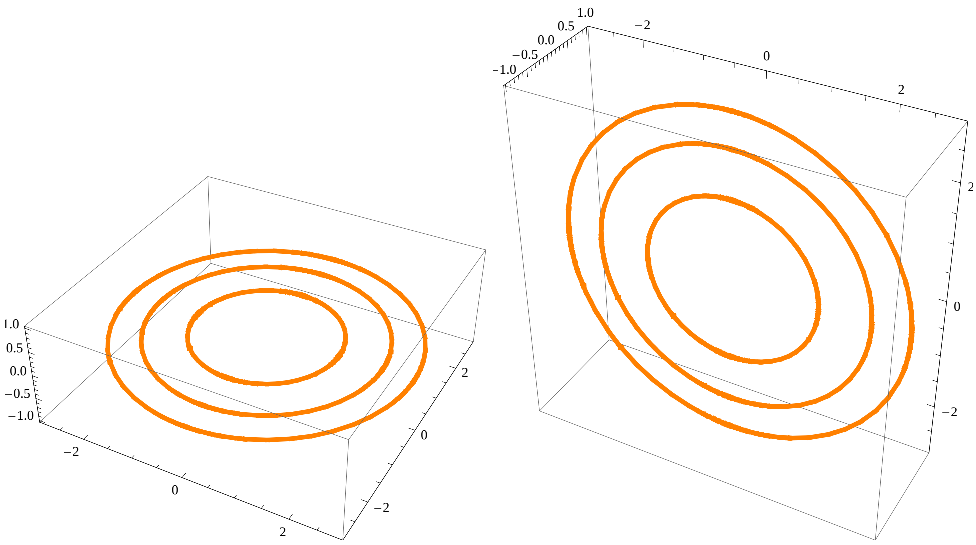

Figure 4: The integral curves of X and Y

and it is the following linear combination of X and Y (in every point of M)

$$[X,Y] = -\frac{z}{x}X + \frac{y}{x}Y. \tag{43}$$

Hence $[X,Y]$ does lies in D and D is involutive. By Frobenius theorem, D is integrable. Each point with $x_0 \neq 0$ lies on a sphere leaf

$$\begin{aligned} \dot{x} &= x_0 \cos\phi \cos\theta + y_0 \sin\theta + z_0 \cos\phi \sin\theta \\ \dot{y} &= -x_0 \sin\phi \cos\theta + y_0 \cos\theta - z_0 \sin\phi \sin\theta \\ \dot{z} &= -x_0 \sin\theta + z_0 \cos\theta; \end{aligned} \tag{44}$$

or, it is trapped on a sphere. [5]

2.7. *Orbits and reachable sets*

Now, we take a set of vector fields X_q , $q = 0, 1, ..., k$ from (20). Let $i_0, \dots, i_q \in \{0, \dots, q\}$. We define the *orbit* $\mathcal{O}(x)$ of x as

$$\mathcal{O}(x) = \left\{ \mathrm{Fl}^{X_{i_q}}_{t^{i_q}}(x) \circ \dots \mathrm{Fl}^{X_{i_0}}_{t^{i_0}}(x); t^{i_0}, \dots, t^{i_q} \in \mathbb{R} \right\}. \tag{45}$$

Further, the *reachable set* $\mathcal{R}(x)$ of x is defined as

$$\mathcal{R}(x) = \left\{ \mathrm{Fl}^{X_{i_q}}_{t^{i_q}}(x) \circ \dots \mathrm{Fl}^{X_{i_0}}_{t^{i_0}}(x); t^{i_0}, \dots, t^{i_q} \geq 0 \right\}, \tag{46}$$

[5]The parametric expression is obtained by the application of the rotation matrix gained as the product of $\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$ onto $[x_0, y_0, z_0]$. Let us realise that we have obtained only incomplete sphere by this rotation: the reverse order of factors (rotation matrices) in the product gives another part of this sphere; a more refined rotation is necessary for the whole sphere.

the *reachable set* $\mathcal{R}(x,\tau)$ *in time* τ of x is defined as

$$\mathcal{R}(x,\tau) = \left\{ \mathrm{Fl}^{X_{i_q}}_{t^{i_q}}(x) \circ \dots \mathrm{Fl}^{X_{i_0}}_{t^{i_0}}(x); t^{i_0}, \dots, t^{i_q} \geq 0, t^{i_0} + \dots + t^{i_q} = \tau \right\} \tag{47}$$

and the *reachable set* $\mathcal{R}_T(x)$ *until time* T of x is defined as

$$\mathcal{R}_T(x) = \bigcup_{\tau \leq T} \mathcal{R}(x,\tau) = \left\{ \mathrm{Fl}^{X_{i_q}}_{t^{i_q}}(x) \circ \dots \mathrm{Fl}^{X_{i_0}}_{t^{i_0}}(x); t^{i_0}, \dots, t^{i_q} \geq 0, t^{i_0} + \dots + t^{i_q} \leq T \right\} \tag{48}$$

There are two notable restrictions of sets above. The first one is a restriction on admissible regulations and the second one is a restriction on trajectories contained in a given neighborhood V of x. Then we write, e.g., $\mathcal{R}^{\mathfrak{U}}_T(x)$, $\mathcal{R}^V_T(x)$, $\mathcal{R}^{\mathfrak{U},V}_T(x)$.

3. The foliations again and more precisely, and moreover the multifoliations

3.1. *Foliations*

We refer to Milnor (1970), Lawson (1974) and Bejancu and Farran (2006) for detailed introductions to the theory of foliations; our adoption is as follows. Let M be a m-dimensional smooth manifold, $m = p + q$, $m \in \mathbb{N}$, $p, q \in \mathbb{N} \cup \{0\}$, $(x, y) = (x^1, \dots, x^p, y^1, \dots, y^q) \in \mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^m$. For constants $\bar{c} \in \mathbb{R}^p$, $c \in \mathbb{R}^q$, we consider spaces $\mathbb{R}^q_{\bar{c}} = \{(x, y) \in \mathbb{R}^m; x^1 = \bar{c}^1, \dots, x^p = \bar{c}^p\}$ and $\mathbb{R}^p_c = \{(x, y) \in \mathbb{R}^m; y^1 = c^1, \dots, y^q = c^q\}$. Intersections of $\mathbb{R}^q_{\bar{c}}$ and $\mathbb{R}^p_c$ with open sets (balls) with respect to the standard topology are denoted by $P^q_{\bar{c}}$ and P^p_c and called the $(\bar{c}, q)$-*coplaque* and the (p, c)-*plaque* in $\mathbb{R}^m$. Suppose that $\mathcal{F} = \{L_t\}_{t \in J}$ is a partition of M into connected subsets, $M = \bigcup_{t \in J} L_t$, $L_t \cap L_s = \emptyset$ for $t \neq s$. Further, we consider a *foliated atlas* on M, i.e., a collection $\{U_i, \varphi_i\}_{i \in I}$, $\varphi_i = \alpha_i \times \beta_i$, $\alpha_i \colon U_i \to \mathbb{R}^p$, $\beta_i \colon U_i \to \mathbb{R}^q$, of charts satisfying

(i) $\{U_i\}_{i \in I}$ is a cover of M by open sets

(ii) each connected component of $L_t \cap U_i$ (for all $i \in I$, $t \in J$) is mapped by φ_i onto an (p, c)-plaque in $\mathbb{R}^m$, i.e., for $u \in U_i$

$$\begin{aligned} x^a &= \alpha^a_i(u) \qquad & a &= 1, \dots, p \\ y^b &= \beta^b_i(u) = c^b \qquad & b &= 1, \dots, q \end{aligned} \tag{49}$$

(iii) transition functions $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$ on $U_i \cap U_j$, $\varphi_{ij} = \alpha_{ij} \times \beta_{ij}$, send (p, c)-plaques onto (p, c)-plaques, i.e.

$$\begin{aligned} x^a &= \alpha^a_{ij}(x, y) \qquad & a &= 1, \dots, p \\ y^b &= \beta^b_{ij}(y) \qquad & b &= 1, \dots, q. \end{aligned} \tag{50}$$

(Maps are regarded as smooth.) Then $\mathcal{F}$ is called the *foliation* of M of *dimension* p and *codimension* q, L_t, $t \in J$ *leaves* of $\mathcal{F}$ and M the *foliated manifold* written shortly by $(M, \mathcal{F})$. Trivial cases arise for $p = 0$, $q = m$ (leaves = points) and for $p = m$, $q = 0$ (the unique leaf = M).

Let $\mathcal{F}$, $\mathcal{F}'$ be two foliations of M with dimensions p and p'. Then $\mathcal{F}'$ is called a *subfoliation* of $\mathcal{F}$ and $\mathcal{F}$ is called a *superfoliation* of $\mathcal{F}'$, denoted by $\mathcal{F}' \preceq \mathcal{F}$, if the following conditions hold:

(i) $0 \le p' \le p \le m$

(ii) for any leaf L' of $\mathcal{F}'$, there exists a leaf L of $\mathcal{F}$ such that $L' \subseteq L$, and the restriction of $\mathcal{F}'$ on a leaf L of $\mathcal{F}$ is a foliation of dimension $p - p'$ of L.

The relation $\preceq$ is an order in the set of foliations of M.

Fibered manifolds are canonically foliated, their fibers can be viewed as leaves. On the other hand, there exist manifolds, which are foliated but not fibered.

3.2. *Transversality of maps, transversality of foliations, multifoliations*

Concepts of a transversality is rather varied, we suggest e.g. Tamura & Sato (1981). Let Δ be an integer greater than 1. Let us consider manifolds H_δ, $\delta = 1, \dots, \Delta$, and M. Let $f_\delta \colon H_\delta \to M$, $\delta = 1, \dots, \Delta$, be (smooth) maps.

We take an arbitrary non-empty subset $E \subseteq \{1, \dots, \Delta\}$ and denote by $\operatorname{Im} f_E$ the intersection of all images of f_ϵ, $\epsilon \in E$.

For $u_E \in \operatorname{Im} f_E$ and every $\epsilon \in E$, let $(Tf_\epsilon)_{u_E}$ denote the image of the tangent map to f_ϵ in u_E; tangent vectors belonging to $(Tf_\epsilon)_{u_E}$ generate a vector subspace of $T_{u_E}M$; we denote it by $\langle (Tf_\epsilon)_{u_E} \rangle$. Further, we denote by $\langle \bigcup_E (Tf_\epsilon)_{u_E} \rangle$ the vector space generated by the union of vectors in all $(Tf_\epsilon)_{u_E}$, $\epsilon \in E$, and by $\langle \bigcap_E (Tf_\epsilon)_{u_E} \rangle$ the vector space generated by vectors belonging to the intersection of all $(Tf_\epsilon)_{u_E}$, $\epsilon \in E$.

For simplicity, we consider only maps for which vector spaces above have constant dimensions for all $u_E \in \operatorname{Im} f_E$.

Now, it is evident that for every chosen $\epsilon_0 \in E$

$$0 \le \dim \langle \bigcap_E (Tf_\epsilon)_{u_E} \rangle \le \dim \langle (Tf_{\epsilon_0})_{u_E} \rangle \le \dim \langle \bigcup_E (Tf_\epsilon)_{u_E} \rangle \le m, \tag{51}$$

or, in the codimension language,

$$m \ge \operatorname{codim} \langle \bigcap_E (Tf_\epsilon)_{u_E} \rangle \ge \operatorname{codim} \langle (Tf_{\epsilon_0})_{u_E} \rangle \ge \operatorname{codim} \langle \bigcup_E (Tf_\epsilon)_{u_E} \rangle \ge 0. \tag{52}$$

Definition 1 Maps $f_\delta \colon H_\delta \to M$, $\delta = 1, \dots, \Delta$, are said to be

$\cap$-*transversal*, if

$$\operatorname{codim} \langle \bigcap_E (Tf_\epsilon)_{u_E} \rangle = \sum_E \operatorname{codim} \langle (Tf_\epsilon)_{u_E} \rangle \tag{53}$$

for all $E \subseteq \{1, \dots, \Delta\}$;

$\cup$-*transversal*, if

$$\sum_E \dim \langle (Tf_\epsilon)_{u_E} \rangle = \dim \langle \bigcup_E (Tf_\epsilon)_{u_E} \rangle \tag{54}$$

for all $E \subseteq \{1, \dots, \Delta\}$.

Remark 1 The Definition 1 implies that f_δ can be $\cap$-transversal only for

$$\sum_{\delta=1}^{\Delta} \operatorname{codim}\langle (Tf_\delta)_{u_{\{1,\dots,\Delta\}}}\rangle \leq m \tag{55}$$

and, analogously, f_δ can be $\cup$-transversal only for

$$\sum_{\delta=1}^{\Delta} \dim\langle (Tf_\delta)_{u_{\{1,\dots,\Delta\}}}\rangle \leq m. \tag{56}$$

It is easy to show that

$$\sum_{\delta=1}^{\Delta} \operatorname{codim}\langle (Tf_\delta)_{u_{\{1,\dots,\Delta\}}}\rangle \leq m \quad \text{and} \quad \sum_{\delta=1}^{\Delta} \dim\langle (Tf_\delta)_{u_{\{1,\dots,\Delta\}}}\rangle \leq m \tag{57}$$

comes into being simultaneously only for $\Delta = 2$ and $\operatorname{codim}\langle (Tf_1)_{u_{\{1,2\}}}\rangle + \operatorname{codim}\langle (Tf_2)_{u_{\{1,2\}}}\rangle = \dim\langle (Tf_1)_{u_{\{1,2\}}}\rangle + \dim\langle (Tf_2)_{u_{\{1,2\}}}\rangle = m$. In this special case, concepts of $\cap$-transversality and $\cup$-transversality are identical.

Let us consider $\cap$-transversal maps $f_\delta\colon H_\delta \to M$, $\delta = 1, \dots, \Delta$ in the following situation: H_δ are subsets (submanifolds) of M and $f_\delta\colon H_\delta \to M$ are their inclusion maps (immersions). Then H_δ are called $\cap$-*transversal*, too. Moreover, if we have Δ foliations F_δ of M, we take in every $u \in M$ their leaves: if they are $\cap$-*transversal* on each choice of u, we say that foliations F_δ of M are $\cap$-*transversal*.

The concept $\cup$-*transversal* foliations F_δ of M comes quite analogously.

Definition 2 A collection $\mathbf{F} = \{\mathcal{F}_\delta\}_{\delta=1}^{\Delta}$ of foliations of M ($\dim M = m$) with dimensions p_δ and codimensions q_δ is called the $\cap$-*multifoliation* ($\cup$-*multifoliation*), if foliations $\mathcal{F}_k$ are $\cap$-transversal ($\cup$-transversal). Especially, the $\cap$-multifoliation ($\cup$-multifoliation) is called *total* $\cap$-*multifoliation* (*total* $\cup$-*multifoliation*) if $\Delta = m$.

Remark 2 It is clear that $q_1 = \dots = q_\Delta = 1$ for total $\cap$-multifoliation and $p_1 = \dots = p_\Delta = 1$ for total $\cup$-multifoliation.

3.3. *Jets modulo multifoliations*

G. Ikegami has defined in his paper (Ikegami 1986) jets modulo foliations. We generalise his concept by the following definition. (In this section, we mean by a multifoliation either $\cap$-multifoliation or $\cup$-multifoliation.)

Definition 3 Let H, M be two manifolds, $f, g\colon H \to M$ maps satisfying $f(h) = g(h) = u \in M$ and let $\mathbf{F} = \{\mathcal{F}_\delta\}_{\delta=1}^{\Delta}$ be a multifoliation of M. Then f is said to have the $(r_1, \dots, r_\Delta)$-*multiorder contact modulo* $\mathbf{F}$ with g at u, if for every Δ-tuple of charts $\left\{U^\delta \ni u, \varphi^\delta\right\}_{1\leq\delta\leq\Delta}$ the maps

$$\alpha^\delta \circ f\colon U^\delta \to \mathbb{R}^{p_\delta} \quad \text{and} \quad \alpha^\delta \circ g\colon U^\delta \to \mathbb{R}^{p_\delta} \tag{58}$$

belong to the same (classical) r_δ-jet at u. (It means that for every curve $\gamma\colon \mathbb{R} \to H$ with $\gamma(0) = h$, the curves $\alpha^\delta \circ f \circ \gamma$ and $\alpha^\delta \circ g \circ \gamma$ have the r_δ-order contact in zero.) As the relation "have the $(r_1, \dots, r_\Delta)$-multiorder contact modulo $\mathbf{F}$" is evidently an equivalence relation, we denote the class of maps having the $(r_1, \dots, r_\Delta)$-multiorder contact modulo $\mathbf{F}$ with f at u by

$$j_h^{r_1,\dots,r_\Delta} f \mod \mathbf{F} \tag{59}$$

and call it $(r_1, \dots, r_\Delta)$*-jet modulo the multifoliation* $\mathbf{F}$ with the *source* $h \in H$ and the *target* $u = f(h) \in M$.

We denote by $J_h^{r_1,\dots,r_\Delta}(H, M; \mathbf{F})_u$ the set of all $(r_1, \dots, r_\Delta)$-jets modulo the multifoliation $\mathbf{F}$ with the h and the target u. Further, we denote

$$J_h^{r_1,\dots,r_\Delta}(H, M; \mathbf{F}) = \bigcup_{u \in M} J_h^{r_1,\dots,r_\Delta}(H, M; \mathbf{F})_u, \tag{60}$$

$$J^{r_1,\dots,r_\Delta}(H, M; \mathbf{F})_u = \bigcup_{h \in H} J_h^{r_1,\dots,r_\Delta}(H, M; \mathbf{F})_u \tag{61}$$

and

$$J^{r_1,\dots,r_\Delta}(H, M; \mathbf{F}) = \bigcup_{u \in M} \bigcup_{h \in H} J_h^{r_1,\dots,r_\Delta}(H, M; \mathbf{F})_u. \tag{62}$$

For manifolds H and M and a multifoliation $\mathbf{F}$ of M, $J_h^{r_1,\dots,r_\Delta}(H, M; \mathbf{F})_u$, $J_h^{r_1,\dots,r_\Delta}(H, M; \mathbf{F})$, $J^{r_1,\dots,r_\Delta}(H, M; \mathbf{F})_u$, and $J^{r_1,\dots,r_\Delta}(H, M; \mathbf{F})$ have a smooth manifold structure. We have bundle projections

$$J^{r_1,\dots,r_\Delta}(H, M; \mathbf{F}) \to H \quad \text{and} \quad J^{r_1,\dots,r_\Delta}(H, M; \mathbf{F}) \to M \tag{63}$$

as well as canonical bundle projections

$$J^{r_1,\dots,r_\Delta}(H, M; \mathbf{F}) \to J^{\tilde{r}_1,\dots,\tilde{r}_\Delta}(H, M; \mathbf{F}) \tag{64}$$

by restricting the multiorder, i.e. for $0 \le \tilde{r}_1 \le r_1, \dots, 0 \le \tilde{r}_\Delta \le r_\Delta$. In doing so

$$J^{0,\dots,0}(H, M; \mathbf{F}) = H \times M. \tag{65}$$

Now, we present that (R, S, Q)-jet are included in the concept of the $(r_1, \dots, r_\Delta)$-jet modulo the multifoliation $\mathbf{F}$. We recall that two morphisms of fibered manifolds determine the same (R, S, Q)-*jet* ($R \le S$, $R \le Q$) at a point y if they have the same R-jet in y, their restrictions to the fiber through y have the same S-jet in y, and their base maps have the same Q-jet in the base point of y.

Let $Y \to M$ be a fibered manifold allowing global sections, $\dim M = q$, $\dim Y = p + q$. The fibered manifold structure of $Y \to M$ determines:

(i) the foliation $\mathcal{F}_1$ with p-dimensional leaves (leaves = fibers)

(ii) the foliation $\mathcal{F}_2$ with q-dimensional leaves (leaves = suitable smooth sections, e.g. for vector bundles can be taken smooth sections including zero section, such as constant smooth sections or something like that); $\mathcal{F}_2$ is non-unique

Thus, we have a (non-unique) multifoliation which is simultaneously $\cup$-multifoliation and $\cap$-multifoliation, see Remark 1. Our construction implies:

Theorem 1 *Let* $\mathbf{F}$ *be a multifoliation given by the fibration as stated above. Then there is a representation of every* (R, S, Q)*-jet as a* (S, Q)*-jet modulo* $\mathbf{F}$.

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